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On lifting q-difference operators in the Askey scheme of basic hypergeometric polynomials

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Abstract
We construct a q-difference operator that lifts the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers up to the continuous big q-Hermite polynomials $H_n(x;a|q)$ on the next level in the Askey scheme of basic hypergeometric polynomials. This operator is defined as Exton's q-exponential function $\varepsilon_q(a_q D_q)$ in terms of the Askey–Wilson divided q-difference operator D_q and it represents a particular q-extension of the standard shift operator $\exp\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right)$. We next show that one can move two steps more upwards in order first to reach the Al-Salam–Chihara family of polynomials $Q_n(x;a,b\,|\,q)$, and then the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$. In both these cases, lifting operators, respectively, turn out to be convolution-type products of two and three one-parameter q-difference operators of the same type $\varepsilon_q(a_q D_q)$ at the initial step. At each step, we also determine q-difference operators that lift the weight function for the continuous q-Hermite polynomials $H_n(x|q)$ successively up to the weight functions for $H_n(x;a|q)$, $Q_n(x;a,b\,|\,q)$ and $p_n(x;a,b,c|\,q)$.

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1. Introduction

To store the current knowledge of a large number of some well-known special functions, scientists decided to construct the so-called Askey scheme of hypergeometric orthogonal polynomials and their q-analogues [1]. Depending on a number of parameters associated with each polynomial family, they occupy different levels in the Askey hierarchy: for instance, the Hermite polynomials $H_n(x)$ are on the ground level, the Laguerre and Charlier polynomials

$L_n^{(\alpha)}(x)$ and $C_n(x;a)$ are one level higher, and so on. Moreover, for some particular or limit values of the parameters, polynomial families from higher levels reduce to those on the lower levels (see [2] and references therein). In other words, one may start with a family at any level in the Askey scheme and then move downwards passing through other known families until he (she) reaches a family on the ground level. The question then naturally arises as to whether there is a possibility of moving in opposite direction, from lower levels to higher ones. Clearly, it would be amounted to determining families of polynomials with a larger number of parameters from initial ones with less number of parameters.

The goal of this work is to examine the possibility of constructing three explicit examples of this type by deriving q-difference operators that successively lift the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers up to the continuous big q-Hermite polynomials $H_n(x;a|\,q)$ in the first step, then to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$ and, finally, to the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ on the higher levels in the Askey scheme of basic hypergeometric polynomials. The building blocks of these operators are Exton's q-exponential functions $\varepsilon_q(a_q D_q)$ in terms of the Askey–Wilson divided q-difference operator D_q.

This paper is organized as follows. Section 2 collects some background facts about the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers, which are then used in section 3 in order to find an explicit form of the q-difference operator $\varepsilon_q(a_q D_q)$ that interrelates $H_n(x|\,q)$ with the continuous big q-Hermite polynomials $H_n(x;a|\,q)$. In section 4, we introduce a convolution-type product operator for two one-parameter q-difference operators $\varepsilon_q(a_q D_q)$ and $\varepsilon_q(b_q D_q)$ and show that this operator lifts the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers up to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$. Section 5 discusses how to build a three-parameter q-difference operator that lifts the continuous q-Hermite polynomials $H_n(x|\,q)$ still one step higher to reach the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$. In sections 3–5, we also explicitly determine q-difference operators that lift the weight function of the continuous q-Hermite $H_n(x|\,q)$ successively up to the weight functions for the continuous big q-Hermite $H_n(x;a|\,q)$, for the Al-Salam–Chihara polynomials $Q_n(x;a,b\,|\,q)$ and for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$. Finally, section 6 concludes this work with a brief discussion of some further research directions of interest.

Throughout this exposition, we employ standard notations of the theory of special functions (see, for example, [1–5]).

2. The ground level: q-Hermite polynomials $H_n(x|\,q)$

The continuous q-Hermite polynomials of Rogers for $0<q<1$ are generated by the three-term recurrence relations

$$H_{n+1}(x|q)=2xH_n(x|q)-(1-q^n)H_{n-1}(x|q), \qquad H_0(x|q)=1, \tag{2.1}$$

and they are orthogonal on the finite interval $-1\leqslant x:=\cos\theta\leqslant 1$:

$$\frac{1}{2\pi}\int_{-1}^{1}H_m(x|q)H_n(x|q)w(x|q)\,\mathrm{d}x=\delta_{mn}e_q(q^{n+1}), \tag{2.2}$$

with respect to the weight function[3]

$$w(x|q):=\frac{1}{\sin\theta}(\mathrm{e}^{2\mathrm{i}\theta};q)_\infty(\mathrm{e}^{-2\mathrm{i}\theta};q)_\infty, \tag{2.3}$$

[3] To avoid any confusion of notations, we note that the weight function $w(x|\,q)$, defined by (2.3) and frequently used in our exposition, is the same as $\widetilde{w}(x|\,q)$ in [1].

where $(a;q)_n$ is the q-shifted factorial, $(a;q)_0 = 1$, $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, $n = 1, 2, 3, \ldots$, and Jackson's q-exponential function $e_q(z)$ is defined as

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = (z;q)_\infty^{-1}, \qquad |z| < 1. \tag{2.4}$$

We also remind the reader of the following Rodrigues-type formula:

$$H_n(x|q)w(x|q) = \left(\frac{q-1}{2}\right)^n q^{n(n-1)/4} D_q^n(w(x|q)), \tag{2.5}$$

for the continuous q-Hermite polynomials $H_n(x|q)$ (see (3.26.10) in [1]). The Rogers generating function for the continuous q-Hermite polynomials $H_n(x|q)$ has the form

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} t^n = e_q(t\,\mathrm{e}^{\mathrm{i}\theta}) e_q(t\,\mathrm{e}^{-\mathrm{i}\theta}), \qquad |t| < 1. \tag{2.6}$$

The polynomials $H_n(x|q)$ satisfy the following q-difference equation:

$$D_q[w(x|q)D_q H_n(x|q)] = \frac{4q(1-q^{-n})}{(1-q)^2} H_n(x|q)w(x|q), \tag{2.7}$$

written in the self-adjoint form (see [1], p 115). The symbol D_q in (2.7) is the conventional notation for the *Askey–Wilson divided q-difference operator* (see, for example, [4], p 529), defined as

$$D_q f(x) := \frac{\delta_q f(x)}{\delta_q x}, \qquad \delta_q g(\mathrm{e}^{\mathrm{i}\theta}) := g(q^{1/2}\,\mathrm{e}^{\mathrm{i}\theta}) - g(q^{-1/2}\,\mathrm{e}^{\mathrm{i}\theta}),$$
$$f(x) \equiv g(\mathrm{e}^{\mathrm{i}\theta}), \qquad x = \cos\theta. \tag{2.8}$$

Merely note that following [6], we find it more convenient for algebraic manipulations to employ the explicit expression

$$D_q = \frac{q^{1/2}}{\mathrm{i}(1-q)} \frac{1}{\sin\theta} (\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} - \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}) = D_{1/q}, \qquad \partial_\theta \equiv \frac{\mathrm{d}}{\mathrm{d}\theta}, \tag{2.9}$$

for D_q (and subsequent q-difference operators) in terms of the shift operators (or the operators of the finite displacement, see [7]) $\mathrm{e}^{\pm z\,\partial_\theta}\, g(\theta) := g(\theta \pm z)$ with respect to the variable θ. In the limit as the deformation parameter $q \uparrow 1$, the Askey–Wilson q-difference operator D_q reduces to the operator of differentiation $\frac{\mathrm{d}}{\mathrm{d}x}$.

Observe that the product rule for the Askey–Wilson operator D_q is known to be of the form

$$D_q(f(x)g(x)) = (\mathcal{A}_q f(x))(D_q g(x)) + (D_q f(x))(\mathcal{A}_q g(x)), \tag{2.10}$$

where $\mathcal{A}_q$ is the so-called *averaging difference operator*, that is (see, for example, [5]),

$$(\mathcal{A}_q f)(x) = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} + \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}) f(x) \equiv \cos(\ln q^{1/2}\partial_\theta) f(x). \tag{2.11}$$

In the limit as $q \uparrow 1$, expression (2.10) reduces to the Newton–Leibniz rule of differentiation for a product of two functions $f(x)$ and $g(x)$.

The Askey–Wilson operator $Q_- \equiv D_q$ represents the lowering operator for the polynomials $H_n(x|q)$ (see formula (3.26.7) in [1])

$$Q_- H_n(x|q) \equiv D_q H_n(x|q) = 2q^{(1-n)/2} \frac{1-q^n}{1-q} H_{n-1}(x|q). \tag{2.12}$$

To verify (2.12), apply the q-difference operator D_q to both sides of the generating function identity (2.6), use the defining relation $(1-z)\,e_q(z) = e_q(qz)$ for the q-exponential function

(2.4) and then equate the coefficients of the same powers of t on both sides. In the following sections, we shall repeatedly use an identity

$$D_q^k H_n(x|q) = \gamma_{n,k}(q) H_{n-k}(x|q), \qquad \gamma_{n,k}(q) := q^{k(k+1)/4} \left(\frac{2q^{-\frac{n}{2}}}{1-q} \right)^k \frac{(q;q)_n}{(q;q)_{n-k}}, \tag{2.13}$$

for integer powers $k = 0, 1, \ldots, n$ of the operator D_q, which is not difficult to deduce from (2.12) by induction on k.

Also note that by a derivation similar to that of (2.12), one has

$$\mathcal{A}_q H_n(x|q) = q^{-n/2}[H_n(x|q) + (q^n - 1)x H_{n-1}(x|q)]. \tag{2.14}$$

There is no difficulty in verifying that the raising operator Q_+ for the polynomials $H_n(x|\,q)$ has the form

$$\begin{aligned} Q_+ \equiv w^{-1}(x|q) D_q w(x|q) &= \frac{\mathrm{i}}{(1-q)\sin\theta} [\mathrm{e}^{2\mathrm{i}\theta} \mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} - \mathrm{e}^{-2\mathrm{i}\theta} \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}] \\ &= q^{-1/2}(1-2x^2) D_q - \frac{4x}{1-q} \mathcal{A}_q, \end{aligned} \tag{2.15}$$

and its action on the polynomials $H_n(x|\,q)$ is (see formula (3.26.9) in [1])

$$Q_+ H_n(x|q) = -\frac{2q^{-n/2}}{1-q} H_{n+1}(x|q). \tag{2.16}$$

It should be pointed out that now (2.16) may also be derived readily from (2.12) and (2.14), upon using the three-term recurrence relation (2.1).

We conclude this section by the following observation about a q-difference equation, which governs the continuous q-Hermite polynomials. In the literature on special functions (see, for example, [1–5]), it is customary to refer to a q-difference equation for the continuous q-Hermite polynomials $H_n(x|\,q)$ only in the self-adjoint form (2.7), which includes the weight function $w(x|\,q)$ associated with them defined in (2.3). Evidently, one may exclude $w(x|\,q)$ from (2.7), by taking into account that

$$\exp(\pm \mathrm{i} \ln q^{1/2} \partial_\theta) w(x|q) = -q^{-1/2}\, \mathrm{e}^{\pm 2\mathrm{i}\theta} w(x|q). \tag{2.17}$$

But as a result, one arrives at a q-difference equation that admits the factorization (details can be found in [6]). This means that the continuous q-Hermite polynomials $H_n(x|\,q)$ actually satisfy a simpler q-difference equation

$$\mathcal{D}_q H_n(x|q) = q^{-n/2} H_n(x|q) \tag{2.18}$$

than the one that follows directly from (2.7) after the elimination of $w(x|\,q)$. The q-difference operator $\mathcal{D}_q$ in (2.18) is defined as (cf (2.9))

$$\mathcal{D}_q = \frac{1}{2\mathrm{i}\sin\theta} (\mathrm{e}^{\mathrm{i}\theta} \mathrm{e}^{\mathrm{i}\ln q^{1/2}\partial_\theta} - \mathrm{e}^{-\mathrm{i}\theta} \mathrm{e}^{-\mathrm{i}\ln q^{1/2}\partial_\theta}), \qquad \partial_\theta \equiv \frac{\mathrm{d}}{\mathrm{d}\theta}, \tag{2.19}$$

and it may be expressed in terms of the q-difference operators D_q and $\mathcal{A}_q$, defined above in (2.9) and (2.11), respectively, as

$$\mathcal{D}_q = \mathcal{A}_q + \frac{1-q}{2q^{1/2}} x D_q.$$

Then the Pearson-type q-difference equation for the weight function $w(x|\,q)$ can be written in the form

$$\mathcal{D}_{1/q} w(x|q) = q^{-1/2} w(x|q), \tag{2.20}$$

which is an easy consequence of relations (2.17) and definition (2.19). The fact that the two q-difference operators in (2.18) and (2.20), which govern the continuous q-Hermite polynomials

$H_n(x|\,q)$ and their orthogonality weight function $w(x|\,q)$, respectively, are interrelated by the formal replacement $q \Rightarrow 1/q$ is very nontrivial because the former q-difference operator $\mathcal{D}_q$ has polynomial (in the independent variable x) eigenfunctions $H_n(x|\,q)$, associated with the discrete spectrum of eigenvalues $q^{-n/2}$, $n = 0, 1, 2, \ldots$, while the latter, $\mathcal{D}_{1/q}$, has a non-polynomial eigenfunction $w(x|\,q)$. This characteristic property of the continuous q-Hermite polynomials and their weight function turns out to be inherited by all lifting operators in this work, and that circumstance will be essentially used for establishing explicit forms of the lifting q-difference operators for the weight functions of other q-polynomial families under discussion. It is important to realize that the factorization of (2.7) in the form (2.18), revealed in [6], represented the decisive step in bringing out this significant link between (2.18) and (2.20).

3. The first level: big q-Hermite polynomials $H_n(x; a|\,q)$

The continuous big q-Hermite polynomials $H_n(x; a|\,q)$ from the next level in the Askey q-scheme for $0 < q < 1$ are generated by the three-term recurrence relations

$$H_{n+1}(x; a|q) = (2x - aq^n)H_n(x; a|q) - (1 - q^n)H_{n-1}(x; a|q), \qquad H_0(x; a|q) = 1, \tag{3.1}$$

and they are explicitly defined by means of the formula

$$H_n(x; a|q) := a^{-n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta} \\ 0, 0 \end{matrix}\middle|\, q; q\right), \qquad x = \cos\theta. \tag{3.2}$$

For real values of the parameter $a \in (-1, 1)$, they are orthogonal on the finite interval $-1 \leqslant x := \cos\theta \leqslant 1$

$$\frac{1}{2\pi}\int_{-1}^{1} H_m(x; a|q)H_n(x; a|q)w(x; a|q)\mathrm{d}x = \delta_{mn}e_q(q^{n+1}), \tag{3.3}$$

with respect to the weight function (cf formula (3.18.2) on p 103 in [1])

$$w(x; a|q) := \frac{1}{\sin\theta}\,\frac{e_q(a\,\mathrm{e}^{\mathrm{i}\theta})e_q(a\,\mathrm{e}^{-\mathrm{i}\theta})}{e_q(\mathrm{e}^{2\mathrm{i}\theta})e_q(\mathrm{e}^{-2\mathrm{i}\theta})} \equiv e_q(a\,\mathrm{e}^{\mathrm{i}\theta})e_q(a\,\mathrm{e}^{-\mathrm{i}\theta})w(x|q), \tag{3.4}$$

where $w(x|\,q)$ is the weight function (2.3) for the continuous q-Hermite polynomials of Rogers $H_n(x|\,q)$. The weight function $w(x; a|\,q)$ in (3.3) enables one to represent $H_n(x; a|\,q)$ in the form of the following Rodrigues-type formula (see (3.18.12) in [1]):

$$H_n(x; a|q)w(x; a|q) = \left(\frac{q-1}{2}\right)^n q^{n(n-1)/4}D_q^n(w(x; q^{n/2}a|q)). \tag{3.5}$$

It should also be recalled that a linear generating function identity for the continuous big q-Hermite polynomials $H_n(x; a|\,q)$ is written as (see (3.18.13) in [1])

$$\sum_{n=0}^{\infty}\frac{H_n(x; a|q)}{(q; q)_n}t^n = (at; q)_\infty e_q(t\mathrm{e}^{\mathrm{i}\theta})e_q(t\,\mathrm{e}^{-\mathrm{i}\theta}), \qquad |t| < 1. \tag{3.6}$$

The action of the Askey–Wilson divided q-difference operator D_q upon the continuous big q-Hermite polynomials $H_n(x; a|\,q)$ does reduce the degree n of the polynomials $H_n(x; a|\,q)$ by 1 (similar to the case of (2.12) from the ground level, see (3.18.9) in [1], p 104), i.e.

$$D_qH_n(x; a|q) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}H_{n-1}(x; q^{1/2}a|q). \tag{3.7}$$

What is important to note about (3.7) is that this action of D_q on $H_n(x; a|\,q)$ also shifts the parameter a to $q^{1/2}a$, so that D_q is actually the *lowering shift operator* with respect to the polynomials $H_n(x; a|\, q)$. As in the case of (2.13), from (3.7), one readily deduces that for integer powers $k = 0, 1, \ldots, n$ of the operator D_q,

$$D_q^k H_n(x; a|q) = \gamma_{n,k}(q) H_{n-k}(x; q^{k/2} a|q), \tag{3.8}$$

where $\gamma_{n,\,k}(q)$ is the same as in (2.13).

The continuous big q-Hermite polynomials $H_n(x; a|\, q)$ can be expressed in terms of the continuous q-Hermite polynomials $H_n(x|\, q)$ of Rogers as

$$H_n(x; a|q) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k H_{n-k}(x|q), \tag{3.9}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{3.10}$$

The coefficients of $H_{n-k}(x|\, q)$ in (3.9) are a special case of the general formula for the connection coefficients of the Askey–Wilson polynomials, derived in [8] (see also [9],[10]). Actually, relation (3.9) enables one to construct an explicit form of the q-difference operator, which lifts the continuous q-Hermite polynomials $H_n(x|\, q)$ up to the next level, $H_n(x; a|\, q)$. Indeed, employing (2.13) on the right-hand side of (3.9), one arrives at the operational formula

$$\begin{aligned} H_n(x; a|q) &= \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-a)^k}{\gamma_{n,k}(q)} D_q^k H_n(x|q) \\ &= \sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/4}}{(q;q)_k} \left(q^{n/2} \frac{(1-q)}{2q} a \right)^k D_q^k H_n(x|q), \end{aligned} \tag{3.11}$$

for the continuous big q-Hermite polynomials (3.2) in terms of the continuous q-Hermite polynomials $H_n(x|\, q)$ of Rogers. Setting $a = q^{-\frac{n}{2}}\, b$, one finally represents (3.11) as

$$H_n(x; q^{-\frac{n}{2}} b|q) = \mathcal{L}_n(b_q D_q|q) H_n(x|q), \qquad b_q := \frac{q-1}{2q} b, \tag{3.12}$$

where $\mathcal{L}_n(b_q\, D_q|\, q)$ is a q-difference operator which consists of a finite sum of integer powers of D_q

$$\mathcal{L}_n(b_q D_q|q) := \sum_{k=0}^{n} c_k(q) (b_q D_q)^k, \qquad c_k(q) := q^{k(k+1)/4}/(q;q)_k. \tag{3.13}$$

It is worth noting that the polynomials

$$\mathcal{L}_n(z|q) = \sum_{k=0}^{n} c_k(q) z^k = c_n(q) z^n {}_3\phi_1(q^{-n/2}, -q^{-n/2}, q^{1/2}; 0; q^{1/2}, q^{n/2}/z)$$

represent the nth partial sum of the power series in z for the Exton q-exponential function $\varepsilon_q(z)$ on the q-linear lattice[4]

$$\mathcal{L}_\infty(z|q) \equiv \varepsilon_q(z) := \sum_{k=0}^{\infty} c_k(q) z^k = {}_1\phi_1(0; -q^{1/2}; q^{1/2}, -q^{1/2} z), \tag{3.14}$$

which was introduced in [11] and has been studied in [12–15].

[4] Observe that we find it more convenient to work with the q-exponential function $\varepsilon_q(z)$ of the form (3.14), which differs from the q-exponential function $\exp_q(z)$ in [3] and [14] by a re-scaling of the argument z, namely, $\varepsilon_q((1-q)z) \equiv \exp_q(q^{1/2} z)$. Since the well-known identity $(q^{-1}; q^{-1})_n = (-1)^n q^{-n(n+1)/2} (q;q)_n$ implies that $c_k(q^{-1}) = (-1)^k c_k(q)$, in our case, $\varepsilon_{1/q}(z) = \varepsilon_q(-z)$, provided that either z does not contain any q-dependent factor or z does include only those q-factors which are symmetric with respect to the replacement $q \Rightarrow 1/q$.

Taking into account that the kth power of the lowering operator D_q annihilates $H_n(x|\,q)$ whenever $k \geqslant n+1$, it is convenient to restate (3.12) in its equivalent form as

$$H_n(x;q^{-\frac{n}{2}}a|q) = \varepsilon_q(a_q D_q)H_n(x|q), \qquad a_q := \frac{q-1}{2q}a. \tag{3.15}$$

We shall call q-difference operators of the type $\mathcal{L}_n(a_q\,D_q|\,q)$ and $\varepsilon_q(a_q\,D_q)$ the *lifting operators*, because their action on q-polynomials increases a number of parameters in the resultant polynomials, which is equivalent to lifting the initial polynomials to higher levels in the Askey scheme of basic orthogonal polynomials. It is to be emphasized that although the action of the operator $\varepsilon_q(a_q\,D_q)$ on the q-Hermite polynomials $H_n(x|\,q)$ is the same as of its truncated counterpart $\mathcal{L}_n(a_q\,D_q|\,q)$ in (3.12), the motivation for writing down (3.15) should be clear: it is actually the entire operator $\varepsilon_q(a_q\,D_q)$ that enables one to construct lifting q-difference operators for such non-polynomial functions in the independent variable x as orthogonality weight functions for q-polynomial families under discussion.

Since the q-exponential function $\varepsilon_q((1-q)\,x)$ reduces to e^x in the limit as $q \uparrow 1$, from (3.13) and (3.14), it is evident that

$$\lim_{q\uparrow 1}\varepsilon_q(a_q D_q) = \exp\left(-\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{3.16}$$

and the operator $\varepsilon_q\,(a_q\,D_q)$ can therefore be regarded as a particular q-extension of the standard shift operator by $-a/2$.

It is well known that in the limit as $q \uparrow 1$, the continuous big q-Hermite polynomials $H_n(x;\,a|\,q)$ reduce to the ordinary Hermite polynomials $H_n(x)$ with shifted argument (see (5.18.2) in [1]):

$$\lim_{q\uparrow 1}\kappa^{-n}H_n(\kappa x; 2\kappa a|q) = H_n(x-a), \qquad \kappa := \sqrt{\frac{1-q}{2}}.$$

In view of (3.16), the same limit follows from (3.15) at once.

Let us also recall here that the q-exponential function $\varepsilon_q(z)$ satisfies the q-difference equation [13, 15]

$$\varepsilon_q(q^{-1/2}z) - \varepsilon_q(q^{1/2}z) = z\varepsilon_q(z), \tag{3.17}$$

which will be essentially used for treating the problem at hand in what follows.

It now remains only to verify, as the consistency check of the approach under discussion, that by applying the appropriate q-difference operator on the three-term recurrence relation (2.1) and the weight function (2.3) for the continuous q-Hermite polynomials $H_n(x|\,q)$, one really obtains the corresponding recurrence relation (3.1) and the weight function (3.4) for the continuous big q-Hermite polynomials $H_n(x;\,a|\,q)$, respectively.

To prove the first part of this statement, we apply to both sides of the recurrence relation (2.1) the q-difference operator $\varepsilon_q(q^{(n+1)/2}\,a_q\,D_q)$. Then, on the left-hand side of (2.1), one gets, by definition (3.15), that

$$\varepsilon_q(q^{(n+1)/2}a_q D_q)H_{n+1}(x|q) = H_{n+1}(x;a|q). \tag{3.18}$$

Let us now evaluate the second term on the right-hand side of (2.1):

$$\begin{aligned}&(1-q^n)\varepsilon_q(q^{(n+1)/2}a_q D_q)H_{n-1}(x|q)\\ &\quad\equiv (1-q^n)\varepsilon_q(q^{(n-1)/2}qa_q D_q)H_{n-1}(x|q) = (1-q^n)H_{n-1}(x;qa|q).\end{aligned} \tag{3.19}$$

Finally, in order to evaluate the first term $\varepsilon_q(q^{\,(n+1)/2}\,a_q\,D_q)(2x\,H_n(x|\,q))$ on the right-hand side of (2.1), it is instructive to first prove the following ancillary operator identity:

$$D_q^k(xH_n(x|q)) = \left[q^{-n/2}\frac{1-q^k}{1-q} + q^{k/2}xD_q\right]D_q^{k-1}H_n(x|q), \qquad k = 1,2,3,\ldots. \tag{3.20}$$

Use the product rule (2.10) and readily verified relations $\mathcal{A}_q\, x = \frac{1}{2}(q^{1/2} + q^{-1/2})x$ and $D_q\, x = 1$ to check that (3.20) is true for $k = 1$. We now assume that (3.20) is valid for some integer value of k and evaluate, by repeatedly employing (2.13), the following:

$$\begin{aligned}
D_q^{k+1}(xH_n(x|q)) &\equiv D_q D_q^k (xH_n(x|q)) \\
&= D_q \left[q^{-n/2}\frac{1-q^k}{1-q} + q^{k/2} x D_q \right] D_q^{k-1} H_n(x|q) \\
&= q^{-n/2}\frac{1-q^k}{1-q} D_q^k H_n(x|q) + q^{k/2} D_q\big(x D_q^k H_n(x|q)\big) \\
&= q^{-n/2}\frac{1-q^k}{1-q} D_q^k H_n(x|q) + q^{k/2}\gamma_{n,k}(q) D_q (xH_{n-k}(x|q)) \\
&= q^{-n/2}\frac{1-q^k}{1-q} D_q^k H_n(x|q) + q^{k/2}\gamma_{n,k}(q)[q^{-(n-k)/2} + q^{1/2} x D_q] H_{n-k}(x|q) \\
&= q^{-n/2}\frac{1-q^k}{1-q} D_q^k H_n(x|q) + q^{k/2}[q^{-(n-k)/2} + q^{1/2} x D_q] D_q^k H_n(x|q) \\
&= q^{-n/2}\left[\frac{1-q^k}{1-q} + q^k\right] D_q^k H_n(x|q) + q^{(k+1)/2} x D_q^{k+1} H_n(x|q) \\
&= \left[q^{-n/2}\frac{1-q^{k+1}}{1-q} + q^{(k+1)/2} x D_q \right] D_q^k H_n(x|q).
\end{aligned} \tag{3.21}$$

This completes the proof of the identity (3.20) by induction on k. □

We are now in a position to evaluate the first term on the right-hand side of (2.1):

$$\begin{aligned}
&\varepsilon_q(q^{(n+1)/2} a_q D_q)(2xH_n(x|q)) = 2\sum_{k=0}^{\infty} c_k(q)[q^{(n+1)/2} a_q D_q]^k (xH_n(x|q)) \\
&= 2xH_n(x|q) + 2\sum_{k=1}^{\infty} c_k(q)[q^{(n+1)/2} a_q]^k D_q^k (xH_n(x|q)) \\
&= 2xH_n(x|q) + 2\sum_{k=1}^{\infty} c_k(q)[q^{(n+1)/2} a_q]^k \left[q^{-n/2}\frac{1-q^k}{1-q} + q^{k/2} x D_q \right] D_q^{k-1} H_n(x|q) \\
&= 2x\left[1 + \sum_{k=1}^{\infty} c_k(q)(q^{n/2} q a_q D_q)^k \right] H_n(x|q) - a\sum_{k=1}^{\infty} c_{k-1}(q)(q^{n/2} q a_q D_q)^{k-1} H_n(x|q) \\
&= (2x-a)\varepsilon_q(q^{n/2} q a_q D_q) H_n(x|q) = (2x-a) H_n(x; qa|q),
\end{aligned} \tag{3.22}$$

where, at the penultimate step, we have used the two-term recurrence relation $(1-q^k)\, c_k(q) = q^{k/2}\, c_{k-1}(q)$ for the coefficients $c_k(q)$ in (2.13). Collecting all three terms (3.18), (3.19) and (3.22), one arrives at the relation

$$H_{n+1}(x; a|q) = (2x-a)H_n(x; qa|q) - (1-q^n)H_{n-1}(x; qa|q), \tag{3.23}$$

which does not quite form a three-term recurrence relation for it actually mixes two big q-Hermite polynomials with distinct parameters a and qa. At this stage, it becomes crucial that the key property (3.17) of the q-exponential function $\varepsilon_q(z)$ turns out to be sufficient for separating them and thus arriving at the three-term recurrence relation with a single parameter. Indeed, by definition (3.15),

$$\varepsilon_q(q^{n/2} q a_q D_q|q) H_n(x|q) = H_n(x; qa|q).$$

On the other hand, from (3.17), with $z = q^{\frac{n+1}{2}} a_q D_q$ and then from (2.12), it follows that

$$\begin{aligned}\varepsilon_q(q^{\frac{n}{2}} q a_q D_q) H_n(x|q) &= [\varepsilon_q(q^{\frac{n}{2}} a_q D_q) - q^{\frac{n+1}{2}} a_q D_q \varepsilon_q(q^{\frac{n+1}{2}} a_q D_q)] H_n(x|q) \\ &= H_n(x; a|q) + a(1-q^n)\varepsilon_q(q^{(n+1)/2} a_q D_q) H_{n-1}(x|q) \\ &= H_n(x; a|q) + a(1-q^n) H_{n-1}(x; qa|q).\end{aligned}$$

Consequently,

$$H_n(x; qa|q) = H_n(x; a|q) + a(1-q^n) H_{n-1}(x; qa|q),$$

and, therefore,

$$H_{n+1}(x; a|q) = H_{n+1}(x; qa|q) - a(1-q^{n+1}) H_n(x; qa|q). \tag{3.24}$$

Upon equating the right-hand sides of (3.23) and (3.24), one finally arrives at the three-term recurrence relation (3.1) for the continuous big q-Hermite polynomials with the parameter qa.

We now shift our attention to the weight functions $w(x|q)$ and $w(x; a|q)$ in the orthogonality relations (2.2) and (3.3) for the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers and the continuous big q-Hermite polynomials $H_n(x; a|q)$, respectively. Our task is to directly verify that not only the polynomials $H_n(x|q)$ and $H_n(x; a|q)$ are interrelated, but their weight functions $w(x|q)$ and $w(x; a|q)$ are connected as well. It is natural to look for such a lifting q-difference operator in the form $\varepsilon_q(\kappa\, a_q\, D_q)$, where $a_q := a\,(q-1)/2\,q$ as in (3.15) and κ is some constant factor, which will be determined from the requirement that $\varepsilon_q\,(\kappa\, a_q\, D_q)\, w(x|q) = w(x; a|q)$. To this end, use the Rodrigues-type formula (2.5) and evaluate, by employing the generating function identity (2.6) for the continuous q-Hermite polynomials $H_n(x|q)$, the following:

$$\begin{aligned}\varepsilon_q(\kappa a_q D_q) w(x|q) &= \sum_{n=0}^{\infty} c_n(q)(\kappa a_q)^n \big(D_q^n w(x|q)\big) \\ &= w(x|q) \sum_{n=0}^{\infty} c_n(q) \left(\frac{2\kappa a_q}{q-1}\right)^n q^{-n(n-1)/4} H_n(x|q) \\ &= w(x|q) \sum_{n=0}^{\infty} \frac{(\kappa a)^n}{q^{n/2}(q;q)_n} H_n(x|q) = e_q(a\,\mathrm{e}^{\mathrm{i}\theta}) e_q(a\,\mathrm{e}^{-\mathrm{i}\theta}) w(x|q) \equiv w(x; a|q),\end{aligned} \tag{3.25}$$

provided that the constant κ is chosen to be equal to $q^{1/2}$. This proves that the q-difference operator $\varepsilon_q(q^{1/2} a_q\, D_q)$ does turn the weight function $w(x|q)$ for the continuous q-Hermite polynomials $H_n(x|q)$ of Rogers into the weight function $w(x; a|q)$ for the continuous big q-Hermite polynomials $H_n(x; a|q)$ in the orthogonality relation (3.3).

We conclude this section with the following remark, which is worthy of attention. Since the identity (3.25) can be rewritten as $\varepsilon_q(a_q\, D_q)\, w(x|q) = w(x; q^{-1/2}a|q)$, it is clear why the additional factor $\kappa = q^{1/2}$ appears in the argument of the Exton q-exponential function in (3.25): the operator $\varepsilon_q(q^{1/2} a_q\, D_q)$ turns the weight function $w(x|q)$ directly into the weight function $w(x; a|q)$, without shifting the parameter a. This means that once the operator $\mathcal{L}_n(a_q\, D_q|q)$ in (3.12) (or, equivalently, the operator $\varepsilon_q(a_q\, D_q)$ in (3.15)), which lifts the continuous q-Hermite polynomials $H_n(x|q)$ up to the continuous big q-Hermite polynomials $H_n(x; a|q)$, is known, one determines an operator, which interrelates the corresponding weight functions $w(x|q)$ and $w(x; a|q)$, by simply adding the factor $q^{1/2}$ into its argument. What should be noted is that the additional factor $q^{1/2}$ makes the operator $\mathcal{L}_n(q^{1/2} a_q D_q|q)$ (and, consequently, the operator $\varepsilon_q(q^{1/2} a_q\, D_q)$) actually symmetric with respect to the change $q \Rightarrow 1/q$ of the base q, that is,

$$\{\mathcal{L}_n(q^{1/2} a_q D_q|q)\}_{q\Rightarrow 1/q} = \mathcal{L}_n(q^{1/2} a_q D_q|q). \tag{3.26}$$

This can be readily verified by bearing in mind that the Askey–Wilson operator D_q is symmetric, i.e. $D_q = D_{1/q}$, whereas the factor $q^{1/2}a_q \equiv (q-1)\,a/2\,q^{1/2}$ changes its sign under the replacement $q \Rightarrow 1/q$ (cf footnote 4). But it is important to stress that such a simple connection between the two q-difference operators, $\mathcal{L}_n(a_q\,D_q|\,q)$ in (3.12) and $\varepsilon_q(q^{1/2}a_q\,D_q)$ in (3.25), exists only in the case of the one-parameter family of q-polynomials $H_n(x;\,a|\,q)$, discussed in this section. As will be clear from the remaining sections, when considering q-polynomial families with two or more parameters, one has to define lifting operators in the form of some convolution-type products of two or more, respectively, one-parameter q-difference operators like $\mathcal{L}_n(a_q\,D_q|\,q)$ and $\varepsilon_q(a_q\,D_q)$. As it will become clear in what follows, these convolution-type q-difference operators turn out to be lacking symmetry with respect to the replacement $q \Rightarrow 1/q$ of the base q, so that to interrelate two or more parametric extensions of the one-parameter, lifting operators $\mathcal{L}_n(a_q\,D_q|\,q)$ and $\varepsilon_q(q^{1/2}a_q\,D_q)$ one step further will be necessary: to replace the base q in the formers by $1/q$.

4. The second level: Al-Salam–Chihara polynomials $Q_n(x; a, b|\,q)$

The Al-Salam–Chihara q-polynomials $Q_n(x;\,a,b|\,q)$ from the next level in the Askey q-scheme depend on two parameters a and b (in addition to the base q) and they are explicitly given as

$$Q_n(x;a,b|q) := \frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n},\, a\,\mathrm{e}^{\mathrm{i}\theta},\, a\,\mathrm{e}^{-\mathrm{i}\theta} \\ ab,\, 0 \end{matrix}\right| q;q\right), \qquad x = \cos\theta. \tag{4.1}$$

For $0 < q < 1$, these polynomials can be generated by the three-term recurrence relations of the form

$$\begin{aligned} &Q_{n+1}(x;a,b|q) = [2x-(a+b)q^n]Q_n(x;a,b|q) - (1-q^n)(1-abq^{n-1})Q_{n-1}(x;a,b|q),\\ &Q_0(x;a,b|q) = 1. \end{aligned} \tag{4.2}$$

If a and b are real or complex conjugates and $\max(|a|,|b|) < 1$, they are orthogonal on the finite interval $-1 \leqslant x := \cos\theta \leqslant 1$:

$$\frac{1}{2\pi}\int_{-1}^{1} Q_m(x;a,b|q)Q_n(x;a,b|q)w(x;a,b|q)\mathrm{d}x = e_q(q^n ab)e_q(q^{n+1})\delta_{mn}, \tag{4.3}$$

with respect to the weight function (cf formula (3.8.2) on p 80 in [1])

$$w(x;a,b|q) := |e_q(a\,\mathrm{e}^{\mathrm{i}\theta})e_q(b\,\mathrm{e}^{\mathrm{i}\theta})|^2 w(x|q), \tag{4.4}$$

where $w(x|\,q)$ is the weight function (2.3) for the continuous q-Hermite polynomials of Rogers $H_n(x|\,q)$. In this work, we assume that the parameters a and b are real and $\max(|a|,|b|) < 1$.

The Rodrigues-type formula for the polynomials $Q_n(x;\,a,b|\,q)$ in terms of the weight function (4.4) is of the form (see (3.8.12) in [1])

$$Q_n(x;a,b|q)w(x;a,b|q) = \left(\frac{q-1}{2}\right)^n q^{n(n-1)/4} D_q^n(w(x;q^{n/2}a,q^{n/2}b|q)). \tag{4.5}$$

It should also be recalled that a linear generating function identity for the Al-Salam–Chihara q-polynomials $Q_n(x;\,a,b|\,q)$ is written as (see (3.8.13) in [1])

$$\sum_{n=0}^{\infty}\frac{Q_n(x;a,b|q)}{(q;q)_n}t^n = (at,bt;q)_\infty e_q(t\mathrm{e}^{\mathrm{i}\theta})e_q(t\mathrm{e}^{-\mathrm{i}\theta}), \qquad |t| < 1, \tag{4.6}$$

where we have used the conventional notation $(a_1,a_2,\ldots,a_k\,;q)_n := \prod_{j=1}^{k}(a_j\,;q)_n$ for products of q-shifted factorials. Similar to the preceding case (3.7), the Askey–Wilson operator D_q is a lowering shift operator for the Al-Salam–Chihara family of q-polynomials $Q_n(x; a, b|\, q)$ of the form (see (3.8.9) in [1])

$$D_q\, Q_n(x; a, b|q) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}\, Q_{n-1}(x; q^{1/2}a, q^{1/2}b|q), \tag{4.7}$$

and an induction argument using (4.7) results in the identity

$$D_q^k\, Q_n(x; a, b|q) = \gamma_{n,k}(q)\, Q_{n-k}(x; q^{k/2}a, q^{k/2}b|q), \tag{4.8}$$

with the same constants $\gamma_{n,\,k}(q)$ as in (2.13).

The Al-Salam–Chihara polynomials $Q_n(x; a, b|\, q)$ are symmetric with respect to the parameters a and b; when one of them vanishes, they reduce to the continuous big q-Hermite polynomials $H_n(x; a|\, q)$, that is, $Q_n(x; a, 0|\, q) = H_n(x; a|\, q)$. For general values of the parameters a and b, these two families are interrelated as [8],[16]

$$Q_n(x; a, b|q) = \sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k H_{n-k}(x; b|q). \tag{4.9}$$

To construct an operator, which lifts the continuous q-Hermite polynomials $H_n(x|\, q)$ up to the Al-Salam–Chihara polynomials $Q_n(x; a, b|\, q)$, it is necessary to first employ (3.8) and then (3.12) on the right-hand side of (4.9), in order to obtain

$$\begin{aligned} Q_n(x; a, b|q) &= \sum_{k=0}^{n} q^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-a)^k}{\gamma_{n,k}(q)} D_q^k H_n(x; q^{-k/2}b|q) \\ &= \sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/4}}{(q;q)_k}\left(q^{n/2}\frac{(1-q)}{2q}a\right)^k D_q^k H_n(x; q^{-k/2}b|q) \\ &= \sum_{k=0}^{n} c_k(q)(q^{n/2}a_q D_q)^k H_n(x; q^{-k/2}b|q) \\ &= \sum_{k=0}^{n} c_k(q)(q^{n/2}a_q D_q)^k \mathcal{L}_n(q^{(n-k)/2}b_q D_q|q) H_n(x|q), \end{aligned} \tag{4.10}$$

where $a_q = (q-1)\,a/2q$ and $b_q = (q-1)\,b/2q$, as before, and the polynomials $\mathcal{L}_n(z|\, q)$ are defined in (3.13). One thus arrives at the required operator form for the Al-Salam–Chihara polynomials $Q_n(x; a, b|\, q)$ in terms of the continuous q-Hermite polynomials $H_n(x|\, q)$ of Rogers:

$$Q_n(x; q^{-n/2}a, q^{-n/2}b|q) = \sum_{k=0}^{n} c_k(q)(a_q D_q)^k \mathcal{L}_n(q^{-k/2}b_q D_q|q) H_n(x|q). \tag{4.11}$$

Naturally, when $b = 0$, this identity exactly coincides with the case (3.12) from the level with one parameter a. Moreover, from the same identity (3.12), one could anticipate that in the case with two parameters, an appropriate lifting q-difference operator should be some composition of two operators, $\varepsilon_q\,(a_q\, D_q)$ and $\varepsilon_q\,(b_q\, D_q)$. But the point is that Exton's q-exponential function $\varepsilon_q(z)$ does not satisfy a simple addition formula $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$ for the ordinary exponential function e^x. Instead, one has (cf formula (14) in a paper by Rahman [14])

$$\varepsilon_q(x)\varepsilon_q(y) = \sum_{n=0}^{\infty} c_n(q) x^n\left(-\frac{y}{x}q^{(1-n)/2}; q\right)_n.$$

Meanwhile, the q-exponential function $\varepsilon_q(x+y)$ can be represented as

$$\varepsilon_q(x+y) = \sum_{n=0}^{\infty} \frac{q^{-n(n-3)/4}}{(q;q)_n}(-x)^n \left(1+\frac{y}{x};q\right)_n \varepsilon_q(q^{-n/2}x), \tag{4.12}$$

with the aid of the inverse Rothe's expansion [17] (see also p 491 in [4])

$$x^n = \sum_{k=0}^{n} (-q^{-n})^k q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (x;q)_k.$$

Identity (4.11) actually determines a product rule of convolution type for the two operators $\mathcal{L}_n(a_q\, D_q|\, q)$ and $\mathcal{L}_n(b_q\, D_q|\, q)$ (and, consequently, for the operators $\varepsilon_q\,(a_q\, D_q)$ and $\varepsilon_q\,(b_q\, D_q)$), which enables one to explicitly construct a q-difference operator, lifting the continuous q-Hermite polynomials $H_n(x|\, q)$ from the ground level to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$ on the second level in the Askey q-scheme. Indeed, let us define a convolution-type symmetric (with respect to the interchange $x \leftrightarrow y$) product of two polynomials $\mathcal{L}_n(x|\,q)$ and $\mathcal{L}_n(y|\,q)$ of the form (cf formula (4.11))

$$\begin{aligned} \mathcal{L}_n(x,y|q) \equiv (\mathcal{L}_n(x|q)\cdot\mathcal{L}_n(y|q))_c &:= \sum_{k=0}^{n} c_k(q) x^k \mathcal{L}_n(q^{-k/2}y|q) \\ &= \sum_{k=0}^{n} c_k(q) x^k \sum_{l=0}^{n} c_l(q)(q^{-k/2}y)^l. \end{aligned} \tag{4.13}$$

In the limit as $n \to \infty$, this definition yields a convolution-type symmetric product of two q-exponential functions $\varepsilon_q(x)$ and $\varepsilon_q(y)$ of the form

$$\begin{aligned} \mathcal{E}_q(x,y) \equiv (\varepsilon_q(x)\cdot\varepsilon_q(y))_c &:= \sum_{n=0}^{\infty} \varepsilon_q^{(n)}(0)\frac{x^n}{n!}\varepsilon_q(q^{-n/2}y) \\ &= \sum_{n=0}^{\infty} c_n(q) x^n \varepsilon_q(q^{-n/2}y), \end{aligned} \tag{4.14}$$

where $\varepsilon_q^{(n)}(0) = \left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,\varepsilon_q(x)\right\}_{x=0}$ and, for notational simplicity in the main results, we have denoted this product as $\mathcal{E}_q(x,y)$, i.e. $\mathcal{E}_q(x,y) \equiv \mathcal{L}_\infty(x,y|\,q)$. One then readily verifies that

$$\begin{aligned} \mathcal{E}_q(a_qD_q, b_qD_q)H_n(x|q) &= \sum_{k=0}^{\infty} c_k(q)(a_qD_q)^k \varepsilon_q(q^{-k/2}b_qD_q)H_n(x|q) \\ &= \sum_{k=0}^{\infty} c_k(q)(a_qD_q)^k \sum_{j=0}^{\infty} c_j(q)(q^{-k/2}b_qD_q)^j H_n(x|q) \\ &= \sum_{k=0}^{n} c_k(q)(a_qD_q)^k \sum_{j=0}^{n} c_j(q)(q^{-k/2}b_qD_q)^j H_n(x|q) \\ &= \sum_{k=0}^{n} c_k(q)(a_qD_q)^k \mathcal{L}_n(q^{-k/2}b_qD_q|q)H_n(x|q) \\ &\equiv \mathcal{L}_n(a_qD_q, b_qD_q|q)H_n(x|q) = Q_n(x;q^{-n/2}a, q^{-n/2}b|q), \end{aligned} \tag{4.15}$$

whilst bearing in mind that the kth power of the lowering operator D_q annihilates $H_n(x|\,q)$ whenever $k \geqslant n+1$ and using the identity (4.11) at the last step.

Once the q-difference operator $\mathcal{E}_q(a_q\, D_q\,, b_q\, D_q)$, lifting the continuous q-Hermite polynomials $H_n(x|\,q)$ up to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$, is explicitly known, it is not difficult to define an associated operator, which lifts the weight function $w(x|\,q)$ for the continuous q-Hermite polynomials (2.3) up to the weight function $w(x;\,a,b|\,q)$ for the Al-Salam–Chihara polynomials (4.4). It turns out that this operator is of the form (cf formula (3.26) in our remark at the end of section 3)

$$\begin{aligned}\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q) &\equiv \mathcal{L}_\infty(q^{1/2}a_q D_q, q^{1/2}b_q D_q|q^{-1})\\ &= \sum_{n=0}^{\infty} c_n(q)(q^{1/2}a_q D_q)^n \varepsilon_q(q^{(n+1)/2}b_q D_q). \qquad (4.16)\end{aligned}$$

To prove this assertion, one evaluates, by first using (2.5) and (2.6), and then (3.5) and (3.6), so that (cf the derivation in (3.25))

$$\begin{aligned}\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q)w(x|q) &= \sum_{k=0}^{\infty} c_k(q)(q^{1/2}a_q D_q)^k \varepsilon_q(q^{(k+1)/2}b_q D_q)w(x|q)\\ &= \sum_{k=0}^{\infty} c_k(q)(q^{1/2}a_q D_q)^k \sum_{j=0}^{\infty} c_j(q)(q^{(k+1)/2}b_q D_q)^j w(x|q)\\ &= \sum_{k=0}^{\infty} c_k(q)(q^{1/2}a_q D_q)^k \left(w(x|q) \sum_{j=0}^{\infty} \frac{H_j(x|q)}{(q;q)_j}(q^{k/2}b)^j \right)\\ &= \sum_{k=0}^{\infty} c_k(q)(q^{1/2}a_q D_q)^k \left(w(x|q)e_q(q^{k/2}b\,\mathrm{e}^{\mathrm{i}\theta})e_q(q^{k/2}b\,\mathrm{e}^{-\mathrm{i}\theta})\right)\\ &= \sum_{k=0}^{\infty} c_k(q)(q^{1/2}a_q D_q)^k w(x;q^{k/2}b|q) = w(x;b|q)\sum_{k=0}^{\infty}\frac{a^k}{(q;q)_k}H_k(x;b|q)\\ &= (ab;q)_\infty w(x;b|q)e_q(a\,\mathrm{e}^{\mathrm{i}\theta})e_q(a\,\mathrm{e}^{-\mathrm{i}\theta}) = (ab;q)_\infty w(x;a,b|q). \qquad (4.17)\end{aligned}$$

Observe that the presence of the constant factor $(ab;q)_\infty$, on the right-hand side of (4.17), only reflects the mathematical fact that the commonly used normalization constants for the weight functions associated with the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers and the Al-Salam–Chihara polynomials $Q_n(x;\,a,b|\,q)$ in the orthogonality relations (2.2) and (4.3), respectively, do not correspond to the same total masses for these two weight functions. That is to say, the lifting q-difference operator $\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q)$ in (4.17) actually eliminates this distinction in the normalization constants by sending $w(x|\,q)$ to the weight function $(ab;q)_\infty\, w(x;\,a,b|\,q)$ with the same value of total mass.

5. The third level: continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$

The continuous dual q-Hahn polynomials $p_n(x;\,a,b,c|\,q)$ from the next level in the Askey q-scheme depend on three parameters a, b and c (in addition to the base q) and they are explicitly given as

$$p_n(x;a,b,c|q) := \frac{(ab,ac;q)_n}{a^n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n}, a\,\mathrm{e}^{\mathrm{i}\theta}, a\,\mathrm{e}^{-\mathrm{i}\theta}\\ ab, ac\end{matrix}\right| q;q\right), \qquad x=\cos\theta. \qquad (5.1)$$

If a, b and c are real, or one of them is real and the other two are complex conjugates and $\max(|\,a|, |\,b|, |\,c|) < 1$), the polynomials $p_n(x;\,a,b,c|\,q)$ are orthogonal on the finite interval

$-1 \leqslant x := \cos\theta \leqslant 1$

$$\frac{1}{2\pi}\int_{-1}^{1} p_m(x;a,b,c|q)\,p_n(x;a,b,c|q)\,w(x;a,b,c|q)\mathrm{d}x = h_n\delta_{mn},$$
$$h_n = (q^{n+1}, abq^n, acq^n, bcq^n; q)_\infty^{-1}, \tag{5.2}$$

with respect to the weight function (cf formula (3.3.2) on p 69 in [1])

$$w(x;a,b,c|q) := |e_q(a\,\mathrm{e}^{\mathrm{i}\theta})e_q(b\,\mathrm{e}^{\mathrm{i}\theta})e_q(c\,\mathrm{e}^{\mathrm{i}\theta})|^2 w(x|q), \tag{5.3}$$

where $w(x|\,q)$ is the weight function (2.3) for the continuous q-Hermite polynomials of Rogers $H_n(x|\,q)$. In what follows we assume that the parameters a, b and c are real and $\max(|\,a|, |\,b|, |\,c|) < 1$.

The Rodrigues-type formula for the polynomials $p_n(x;a,b,c|\,q)$ in terms of the weight function (5.3) is (see (3.3.12) in [1])

$$p_n(x;a,b,c|q)w(x;a,b,c|q) = \left(\frac{q-1}{2}\right)^n q^{n(n-1)/4} D_q^n w(x; q^{\frac{n}{2}}a, q^{\frac{n}{2}}b, q^{\frac{n}{2}}c|q). \tag{5.4}$$

In a similar vein as in the two preceding cases (3.7) and (4.7), the Askey–Wilson divided q-difference operator D_q is a lowering shift operator for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ (see (3.3.9) in [1]), i.e.

$$D_q p_n(x;a,b,c|q) = 2q^{(1-n)/2}\frac{1-q^n}{1-q}p_{n-1}(x;q^{1/2}a, q^{1/2}b, q^{1/2}c|q). \tag{5.5}$$

Consequently, as in all those cases,

$$D_q^k p_n(x;a,b,c|q) = \gamma_{n,k}(q)\,p_{n-k}(x;q^{k/2}a, q^{k/2}b, q^{k/2}c|q), \qquad k = 0, 1, 2, \ldots, \tag{5.6}$$

with the same constants $\gamma_{n,\,k}(q)$ as in (2.13).

The continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ are symmetric with respect to the parameters a, b and c; when one of them is equated to zero, they reduce to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$, that is, $p_n(x;a,b,0|\,q) = Q_n(x;a,b|\,q)$. For general values of the parameters a, b and c, these two families are interrelated as [8]

$$p_n(x;a,b,c|q) = \sum_{k=0}^{n} C_{n,k}\,Q_k(x;b,c|q), \tag{5.7}$$

where the connection coefficients

$$C_{n,k} = q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_q c^{k-n}\frac{(ac,bc;q)_n}{(ac,bc;q)_k}\,{}_2\phi_1\left(\left.\begin{matrix} q^{k-n}, 0 \\ acq^k \end{matrix}\right| q;q\right). \tag{5.8}$$

The basic ${}_2\phi_1$-polynomial on the right-hand side of (5.8) can be evaluated as a special case of the Chu–Vandermonde q-sum for ${}_2\phi_1(q^{-n}, b\,; c\,; q, q)$ with a vanishing parameter b ([3], formula (1.5.3) on p 14, see also [18]), that is,

$${}_2\phi_1(q^{-n}, 0; c; q, q) = q^{n(n-1)/2}\frac{(-c)^n}{(c;q)_n}.$$

Thus, (5.7) reduces to the relation

$$p_n(x;a,b,c|q) = \sum_{k=0}^{n} q^{(n-k)(n-k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k}\frac{(bc;q)_n}{(bc;q)_k}Q_k(x;b,c|q)$$
$$\equiv \sum_{m=0}^{n} q^{m(m-1)/2}\begin{bmatrix} n \\ m \end{bmatrix}_q (-a)^m\frac{(bc;q)_n}{(bc;q)_{n-m}}Q_{n-m}(x;b,c|q)$$

$$= \sum_{m=0}^{n} (abcq^{n-1})^m \left(\frac{q^{1-n}}{bc}; q\right)_m \begin{bmatrix} n \\ m \end{bmatrix}_q Q_{n-m}(x; b, c|q)$$

$$= \sum_{m=0}^{n} q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q (-a)^m (bcq^{n-1}; q^{-1})_m Q_{n-m}(x; b, c|q), \tag{5.9}$$

where, at the last step, we employ the inversion formula

$$(z; q^{-1})_n = q^{-n(n-1)/2}(-z)^n (z^{-1}; q)_n,$$

with respect to the transformation of the base $q \Rightarrow 1/q$.

To construct an operator, which lifts the continuous q-Hermite polynomials $H_n(x|q)$ up to the continuous dual q-Hahn polynomials $p_n(x; a, b, c|q)$, it is necessary to first use (4.8) and then (4.10) on the right-hand side of (5.9), in order to obtain that

$$p_n(x; a, b, c|q) = \sum_{m=0}^{n} \frac{q^{m(m-1)/2}}{\gamma_{n,m}(q)} \begin{bmatrix} n \\ m \end{bmatrix}_q (-a)^m (q^{n-1}bc; q^{-1})_m D_q^m Q_n(x; q^{-\frac{m}{2}}b, q^{-\frac{m}{2}}c|q)$$

$$= \sum_{m=0}^{n} (bcq^{n-1}; q^{-1})_m c_m(q) (q^{n/2} a_q D_q)^m Q_n(x; q^{-m/2}b, q^{-m/2}c|q)$$

$$= \sum_{m=0}^{n} (bcq^{n-1}; q^{-1})_m c_m(q) (q^{n/2} a_q D_q)^m$$

$$\times \sum_{k=0}^{n} c_k(q) (q^{(n-m)/2} b_q D_q)^k \mathcal{L}_n(q^{(n-m-k)/2} c_q D_q|q) H_n(x|q). \tag{5.10}$$

Consequently, in the same spirit as in the preceding cases (3.12) and (4.14), we rewrite (5.10) in the operator form as

$$p_n(x; q^{-n/2}a, q^{-n/2}b, q^{-n/2}c|q) = \mathcal{L}_n(a_q D_q, b_q D_q, c_q D_q|q) H_n(x|q), \tag{5.11}$$

where $a_q = (q-1)a/2q$, $b_q = (q-1)b/2q$, $c_q = (q-1)c/2q$ and the q-difference operator $\mathcal{L}_n(a_q\, D_q,\, b_q\, D_q,\, c_q\, D_q|\, q)$ is defined as (cf definition (4.13))

$$\mathcal{L}_n(a_q D_q, b_q D_q, c_q D_q|q)$$

$$:= \sum_{k=0}^{n} c_k(q) (bc/q; q^{-1})_k (a_q D_q)^k \mathcal{L}_n(q^{-k/2} b_q D_q, q^{-k/2} c_q D_q|q). \tag{5.12}$$

Observe that for the zero values of any of the three parameters a, b and c, the operator identity (5.11) reduces to the case of the Al-Salam–Chihara polynomials (see the last line in (4.15)) with remaining two non-zero parameters.

Now taking into account that the kth power of the lowering operator D_q annihilates $H_n(x|\,q)$ whenever $k \geqslant n+1$, one can extend the upper limits in all internal sums in (5.11) from n to infinity. Thus, one arrives at the final desired operator form of the relation between the continuous dual q-Hahn polynomials $p_n(x; a, b, c|\,q)$ and the continuous q-Hermite polynomials $H_n(x|\,q)$:

$$p_n(x; q^{-n/2}a, q^{-n/2}b, q^{-n/2}c|q) = \mathcal{E}_q(a_q D_q, b_q D_q, c_q D_q) H_n(x|q), \tag{5.11$'$}$$

where, by definition,

$$\mathcal{E}_q(a_q D_q, b_q D_q, c_q D_q) \equiv \mathcal{L}_\infty(a_q D_q, b_q D_q, c_q D_q|q)$$

$$= \sum_{k=0}^{\infty} \left(\frac{bc}{q}; q^{-1}\right)_k c_k(q) (a_q D_q)^k \mathcal{E}_q(q^{-k/2} b_q D_q, q^{-k/2} c_q D_q). \tag{5.13}$$

Once the q-difference operator $\mathcal{L}_n(a_q\, D_q, b_q\, D_q, c_q\, D_q|\, q)$ (or its equivalent operator $\mathcal{E}_q(a_q\, D_q, b_q\, D_q, c_q\, D_q)$, which lifts the q-Hermite polynomials $H_n(x|\, q)$ up to the dual q-Hahn polynomials $p_n(x; a, b, c|\, q)$, is explicitly defined, from the previous cases, we now know how to determine an associated operator with it, which lifts the weight function $w(x|\, q)$ for the q-Hermite polynomials (2.3) up to the weight function $w(x; a, b, c|\, q)$ for the dual q-Hahn polynomials (5.3). Indeed, the appropriate q-difference operator is (cf q-difference operator (4.16))

$$\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q, q^{1/2}c_q D_q) \equiv \mathcal{L}_\infty(q^{1/2}a_q D_q, q^{1/2}b_q D_q, q^{1/2}c_q D_q|q^{-1}). \tag{5.14}$$

To prove this assertion, one evaluates, by using (4.17) in the first step and then the well-known property $e_q(q^n\, z) = (z; q)_n\, e_q(z)$ of Jackson's q-exponential function $e_q(z)$, that

$$\begin{aligned}
&\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q, q^{1/2}c_q D_q)w(x|q)\\
&\quad= \sum_{m=0}^{\infty} c_m(q)(bc; q)_m (q^{1/2}a_q D_q)^m \mathcal{E}_{1/q}(q^{(m+1)/2}b_q D_q, q^{(m+1)/2}c_q D_q)w(x|q)\\
&\quad= \sum_{m=0}^{\infty} c_m(q)(bc; q)_m (bcq^m; q)_\infty (q^{1/2}a_q D_q)^m w(x; q^{m/2}b, q^{m/2}c|q)\\
&\quad= (bc; q)_\infty \sum_{m=0}^{\infty} c_m(q)(q^{1/2}a_q)^m D_q^m w(x; q^{m/2}b, q^{m/2}c|q)\\
&\quad= (bc; q)_\infty w(x; b, c|q) \sum_{m=0}^{\infty} \frac{a^m}{(q; q)_m} Q_m(x; b, c|q)\\
&\quad= (ab, ac, bc; q)_\infty w(x; b, c|q) e_q(a\, \mathrm{e}^{\mathrm{i}\theta}) e_q(a\, \mathrm{e}^{-\mathrm{i}\theta})\\
&\quad= (ab, ac, bc; q)_\infty w(x; a, b, c|q),
\end{aligned} \tag{5.15}$$

upon employing (4.6) at the last two steps, followed by (5.3).

As in the relation (4.17) between weight functions $w(x|\, q)$ and $w(x; a, b|\, q)$, the presence of the constant factor on the right-hand side of (5.15) is consistent with the fact that the weight function $w(x; a, b, c|\, q)$ is commonly normalized in the orthogonality relation (5.2) to the constant $h_0 = (ab, ac, bc, q; q)_\infty^{-1}$; meanwhile, the normalization constant for $w(x|\, q)$ in (2.2) is $e_q(q) \equiv (q; q)_\infty^{-1}$; thus, the constant factor on the right-hand side of (5.15) is exactly the ratio $e_q(q)/h_0$. That is to say, (5.15) can simply be restated as

$$\mathcal{E}_{1/q}(q^{1/2}a_q D_q, q^{1/2}b_q D_q, q^{1/2}c_q D_q)w^{(ren)}(x|q) = w^{(ren)}(x; a, b, c|q), \tag{5.16$'$}$$

provided that the weight functions in this identity are normalized in such a way that total masses for both of them are now equal to 2π, that is, $w^{(ren)}(x|\, q) := (q; q)_\infty\, w(x|\, q)$ and $w^{(ren)}(x; a, b, c|\, q) := (ab, ac, bc, q\,; q)_\infty\, w(x; a, b, c|\, q)$.

6. Concluding comments and outlook

To summarize, we have explicitly determined the q-difference operators that provide complete (i.e. which includes weight functions associated with these polynomials) lift from the continuous q-Hermite polynomials $H_n(x|\, q)$ of Rogers successively up to first reach the continuous big q-Hermite polynomials $H_n(x; a|\, q)$, then the Al-Salam–Chihara polynomials $Q_n(x; a, b|\, q)$ and, finally, the continuous dual q-Hahn polynomials $p_n(x; a, b, c|\, q)$ on the next three levels in the Askey scheme of basic hypergeometric polynomials.

It has been a part of folklore in the theory of special functions that polynomial families on the lowest (ground) levels in the Askey hierarchy seem to predetermine properties of polynomials on all higher levels. For a concrete manifestation in favour of such opinion, we refer to a paper by Berg and Ismail [19], who formulated a procedure of attaching generating functions to orthogonality measures. By using this procedure, one may start with the weight function for the continuous q-Hermite polynomials $H_n(x\,|\,q)$ of Rogers and then climb up in the Askey q-scheme by reaching the weight functions first for Al-Salam–Chihara polynomials and then for the most general case in this scheme, Askey–Wilson polynomials. It seems that we have considered a more general approach in this work, which enables one to interconnect not only weight functions in the Askey q-scheme (as in [19]), but also polynomials themselves.

Of course, it would be of considerable interest to show that by using the same approach, one can also reach the Askey–Wilson polynomials on the top level in the Askey q-scheme. But the point is that in attempting to do so, one is confronted by the necessity of evaluating an intricate non-standard generating function for the continuous dual q-Hahn polynomials $p_n(x; a, b, c\,|\,q)$. We are currently searching for a way to overcome this purely technical difficulty.

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